

Limits, Derivatives, and Integrals

Assumptions

All functions, variables and constants are real valued.

f is a function. v and u are functions of a common independent variable.

x is a real valued variable; a, b, c, d and n are real number constants.

Limits

$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$	$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$	$\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$	$\lim_{x \rightarrow 0} \left(1 + \frac{1}{x}\right)^x = 1$	$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x \triangleq e$
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Derivative Definition and Properties

Definition	General: $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$	At the point a : $f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$
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Properties	$(cu)' = cu'$	$(u \pm v)' = u' \pm v'$	$(uv)' = u'v + uv'$	$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$
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Chain Rule	$[f(u(x))]' = f'(u(x))u'(x)$
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L'Hôpital's Rule

Suppose f and g are differentiable functions on an open interval containing a , but not necessarily at a , and

$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0$ or $\pm \infty$ and in the interval $g'(x) \neq 0$ except possibly at a , if $\lim_{x \rightarrow a} [f'(x)/g'(x)]$ exists, or is $\pm \infty$ then

$\lim_{x \rightarrow a} f(x)/g(x) = \lim_{x \rightarrow a} f'(x)/g'(x)$. Moreover, this holds for the limits: $x \rightarrow a^-$, $x \rightarrow a^+$ or $x \rightarrow \pm \infty$.

Derivative of Basic Functions

If x is replaced with a function in the derivatives below *the chain rule must be applied*.

Constants, Power, Exponential and Logarithmic Functions

$c' = 0$	$(x^n)' = nx^{n-1}$ (1)	$(e^x)' = e^x$	$(b^x)' = (\ln b)b^x$ $b > 0$	$(\ln x)' = \frac{1}{x}$, $x \neq 0$	$(\log_b x)' = \frac{1}{x \ln b}$, $x \neq 0$, $b > 0$
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Trigonometric Functions

$(\sin x)' = \cos x$	$(\cos x)' = -\sin x$
$(\tan x)' = \sec^2 x$	$(\cot x)' = -\csc^2 x$
$(\sec x)' = \sec x \tan x$	$(\csc x)' = -\csc x \cot x$

Inverse Trigonometric Functions

$(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}$	$(\cos^{-1} x)' = \frac{-1}{\sqrt{1-x^2}}$
$(\tan^{-1} x)' = \frac{1}{1+x^2}$	$(\cot^{-1} x)' = \frac{-1}{1+x^2}$
$(\sec^{-1} x)' = \frac{1}{ x \sqrt{x^2-1}}$	$(\csc^{-1} x)' = \frac{-1}{ x \sqrt{x^2-1}}$

(1) For $x \geq 0$ and $n \in \mathbb{Z}$ and $\neq 0$ or $x < 0$ and $n \in \mathbb{Z}$ in lowest terms with a positive odd denominator.

Functions as Limits of Integration

Given functions f , g and h are integrable, and function F is an antiderivative of f the Fundamental Theorem of Calculus tells us

$$\int_{h(x)}^{g(x)} f(t) dt = \int_0^{g(x)} f(t) dt - \int_0^{h(x)} f(t) dt = F(g(x)) - F(0) - (F(h(x)) - F(0)) = F(g(x)) - F(h(x)).$$

Applying the chain rule of differentiation gives,

$$\frac{d}{dx} \int_{h(x)}^{g(x)} f(t) dt = \frac{d}{dx} [F(g(x)) - F(h(x))] = \frac{d}{dx} [F(g(x))] - \frac{d}{dx} [F(h(x))] = f(g(x))g'(x) - f(h(x))h'(x).$$

Integrals of Basic Functions

Constant, Power, Exponential, 1/x, and Logarithmic Functions

$\int c dx = cx + C$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$	$\int \frac{1}{x} dx = \ln x + C, x \neq 0$	$\int e^x dx = e^x + C$
$\int b^x dx = \frac{b^x}{\ln b} + C, b > 0$	$\int \ln x dx = x \ln x - x + C, x > 0$	$\int \log_b x dx = \frac{1}{\ln b} (x \ln x - x) + C, x > 0, b > 0$	

Trigonometric Functions

$\int \sin x dx = -\cos x + C$	$\int \tan x dx = \ln \sec x + C$	$\int \csc x dx = \ln \csc x - \cot x + C$ $= \ln \tan(x/2) + C$
$\int \cos x dx = \sin x + C$	$\int \cot x dx = \ln \sin x + C$	$\int \sec x dx = \ln \sec x + \tan x + C$ $= \ln \tan(\pi/4 + x/2) + C$

Inverse Trigonometric Functions

$\int \sin^{-1} x dx = x \sin^{-1} x + \sqrt{1-x^2} + C$	$\int \cos^{-1} x dx = x \cos^{-1} x - \sqrt{1-x^2} + C$
$\int \tan^{-1} x dx = x \tan^{-1} x - \ln \sqrt{x^2+1} + C$	$\int \cot^{-1} x dx = x \cot^{-1} x + \ln \sqrt{x^2+1} + C$
$\int \sec^{-1} x dx = x \sec^{-1} x - \ln x + \sqrt{x^2-1} + C$	$\int \csc^{-1} x dx = x \csc^{-1} x + \ln x + \sqrt{x^2-1} + C$

Some Functions containing $x^2 \pm a^2$ and $a^2 - x^2$

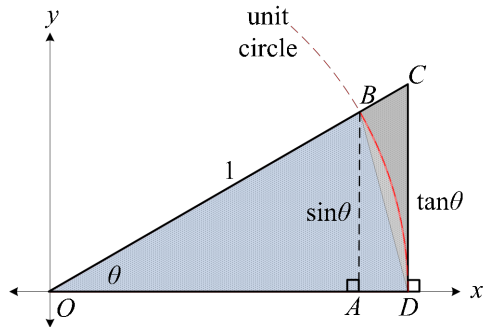
$\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln x + \sqrt{x^2-a^2} + C$	$\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln(x + \sqrt{x^2+a^2}) + C$	$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1} \frac{x}{a} + C = -\cos^{-1} \frac{x}{a} + C$
$\int \frac{1}{x^2-a^2} dx = \frac{1}{2a} \ln \left \frac{x-a}{x+a} \right + C$	$\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C = -\frac{1}{a} \cot^{-1} \frac{x}{a} + C$	
$\int \sqrt{x^2-a^2} dx = \frac{x}{2} \sqrt{x^2-a^2} - \frac{1}{2} a^2 \ln x + \sqrt{x^2-a^2} + C$	$\int \sqrt{a^2-x^2} dx = \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{x}{2} \sqrt{a^2-x^2} + C$	

Trigonometric Reductions

$\int \sin^n x dx = \frac{-1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x dx$	$\int \cos^n x dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x dx$
$\int \tan^n x dx = \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x dx$	$\int \cot^n x dx = \frac{-1}{n-1} \cot^{n-1} x - \int \cot^{n-2} x dx$
$\int \csc^n x dx = \frac{-1}{n-1} \csc^{n-2} x \cot x + \frac{n-2}{n-1} \int \csc^{n-2} x dx$	$\int \sec^n x dx = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x dx$

Proof of Limits

Proof $\lim_{x \rightarrow 0} \frac{\sin x}{x}$.



In the diagram to the left quadrant I of the unit circle is shown with two triangles and a sector drawn on the same angle θ . Since it is a unit circle the lengths $OB = OD = 1$. The blue triangle $\triangle OBD$ has area $(\sin \theta)/2$. The dotted triangle $\triangle OCD$ has area $(\tan x)/2$. The sector formed by the arc BD has area $\pi \theta / (2\pi) = \theta/2$

The area of $\triangle OBD$ is a subset of the area of the sector which in turn is a subset of the area of $\triangle OCD$. This allows us to write,

$$\frac{\sin \theta}{2} \leq \frac{\theta}{2} \leq \frac{\tan \theta}{2}.$$

We need to consider a negative θ in quadrant IV so we can take the two-sided limit as $\theta \rightarrow 0$. Since area is taken to be always positive, we add absolute value signs to each

area to ensure the area remains positive in quadrant IV,

$$\left| \frac{\sin \theta}{2} \right| \leq \left| \frac{\theta}{2} \right| \leq \left| \frac{\tan \theta}{2} \right|$$

Multiplying this equation by $|2/\sin \theta|$ gives,

$$1 \leq \left| \frac{\theta}{\sin \theta} \right| \leq \left| \frac{1}{\cos \theta} \right|.$$

In quadrant I and IV $\cos \theta$ is positive and both θ and $\sin \theta$ have the same sign. This means all terms in the inequality are positive and the absolute value signs can be removed,

$$1 \leq \frac{\theta}{\sin \theta} \leq \frac{1}{\cos \theta}.$$

Taking the reciprocal, we reverse the inequality signs.

$$1 \geq \frac{\sin \theta}{\theta} \geq \cos \theta$$

Apply the squeeze theorem with the limit $x \rightarrow 0$.

$$\lim_{x \rightarrow 0} 1 \geq \lim_{x \rightarrow 0} \frac{\sin x}{x} \geq \lim_{x \rightarrow 0} \cos x$$

$$1 \geq \lim_{x \rightarrow 0} \frac{\sin x}{x} \geq 1$$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

□

Proof $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$.

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} &= \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} \cdot \frac{1 + \cos x}{1 + \cos x} \\ &= \lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{x(1 + \cos x)} \\ &= \lim_{x \rightarrow 0} \frac{\sin^2 x}{x(1 + \cos x)} \\ &= \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \frac{\sin x}{1 + \cos x} \\ &= \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \lim_{x \rightarrow 0} \frac{\sin x}{1 + \cos x} \\ &= 1 \cdot 0 \\ &= 0 \\ &\square \end{aligned}$$

Proof $\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$.

Use the definition

$$e \triangleq \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x \rightarrow e = \lim_{x \rightarrow 0} (1+x)^{1/x}$$

to write,

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{e^x - 1}{x} &= \lim_{x \rightarrow 0} \frac{\left(\lim_{x \rightarrow 0} (1+x)^{1/x}\right)^x - 1}{x} \\ &= \lim_{x \rightarrow 0} \frac{\left((1+x)^{1/x}\right)^x - 1}{x} \\ &= \lim_{x \rightarrow 0} \frac{(1+x)^{x/x} - 1}{x} \\ &= \lim_{x \rightarrow 0} \frac{1 + x - 1}{x} \\ &= \lim_{x \rightarrow 0} \frac{x}{x} \\ &= \lim_{x \rightarrow 0} 1 \\ &= 1 \\ &\square \end{aligned}$$

Proof $\lim_{x \rightarrow 0} \left(1 + \frac{1}{x}\right)^x = 1$.

$$\lim_{x \rightarrow 0} \left(1 + \frac{1}{x}\right)^x = e^{\ln\left(\lim_{x \rightarrow 0} \left(1 + \frac{1}{x}\right)^x\right)}$$

Focusing on the exponent on the right above and utilizing L'Hôpital's rule we have,

$$\begin{aligned} \ln\left(\lim_{x \rightarrow 0} \left(1 + \frac{1}{x}\right)^x\right) &= \lim_{x \rightarrow 0} \ln\left(1 + \frac{1}{x}\right)^x \\ &= \lim_{x \rightarrow 0} x \ln\left(1 + \frac{1}{x}\right) \\ &= \lim_{x \rightarrow 0} \frac{\ln\left(1 + \frac{1}{x}\right)}{\frac{1}{x}} \\ &= \lim_{x \rightarrow 0} \frac{\frac{d}{dx}\left(\ln\left(1 + \frac{1}{x}\right)\right)}{\frac{d}{dx}\left(\frac{1}{x}\right)} \\ &= \lim_{x \rightarrow 0} \frac{\left(\frac{1}{1+1/x}\right)\left(\frac{-1}{x^2}\right)}{\frac{-1}{x^2}} \\ &= \lim_{x \rightarrow 0} \frac{1}{1+1/x} \\ &= 0. \end{aligned}$$

Replace this result for the exponent in the original equation,

$$\begin{aligned} \lim_{x \rightarrow 0} \left(1 + \frac{1}{x}\right)^x &= e^0 \\ &= 1 \\ &\square \end{aligned}$$

Proof of Derivatives

Derivatives of Power Function

For $n \in \mathbb{Z}$ and $n > 0$:

$$\begin{aligned} \frac{d}{dx} x^n &= \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h}, \text{ use binomial expansion for } (x+h)^n \text{ to write} \\ &= \lim_{h \rightarrow 0} \frac{x^n + \binom{n}{1} x^{n-1} h + \binom{n}{2} x^{n-2} h^2 + \dots + \binom{n}{n-1} x h^{n-1} + h^n - x^n}{h} \\ &= \lim_{h \rightarrow 0} \left(\binom{n}{1} x^{n-1} + \binom{n}{2} x^{n-2} h + \dots + \binom{n}{n-1} x h^{n-2} + h^{n-1} \right) \\ &= \binom{n}{1} x^{n-1} \\ &= n x^{n-1} \\ &\square \end{aligned}$$

For $n \in \mathbb{Z}$ and $n < 0$:

Let $m = -n$ so $m > 0$ and the previous result can be used along with the quotient rule:

$$\begin{aligned} x^n &= x^{-m} = \frac{1}{x^m} \\ \frac{d}{dx} x^n &= \frac{0 \cdot x^m - m x^{m-1} \cdot 1}{x^{2m}} \\ &= -m x^{m-1-2m} \\ &= -m x^{-m-1} \\ &= n x^{n-1} \\ &\square \end{aligned}$$

For $r \in \mathbb{R}$ and $x > 0$,

$$\begin{aligned} y &= x^r \\ \ln y &= \ln x^r \\ \ln y &= r \ln x \\ \frac{1}{y} \frac{dy}{dx} &= r \frac{1}{x} \\ \frac{dy}{dx} &= \frac{r \cdot y}{x} \\ &= \frac{r \cdot x^r}{x} \\ &= r x^{r-1} \\ &\square \end{aligned}$$

For $r = p/q \in \mathbb{Q}$ where p and q are in lowest terms and q is odd, then x^r is defined for all $x < 0$. Below we show that the power rule for differentiation still hold under these conditions.

We can rewrite $x^{\frac{p}{q}} = ((-1)(-x))^{\frac{p}{q}} = (-1)^{\frac{p}{q}} (-x)^{\frac{p}{q}}$. This can be decomposed into two cases depending on the parity of p and using the property that for even p we have $(-1)^{\frac{p}{q}} = 1$ and for odd p we have $(-1)^{\frac{p}{q}} = -1$

$$\text{For even } p, x^{\frac{p}{q}} = (-1)^{\frac{p}{q}} (-x)^{\frac{p}{q}} = (-x)^{\frac{p}{q}}$$

$$\text{For odd } p, x^{\frac{p}{q}} = (-1)^{\frac{p}{q}} (-x)^{\frac{p}{q}} = -(-x)^{\frac{p}{q}}$$

With $x < 0$ the base of the exponent, $-x$, is positive, and we can apply our previously established rule for positive powers together with the chain rule.

With $y = x^{\frac{p}{q}}$ and using the property that the difference of an even and an odd is odd so $(-1)^{\frac{p-q}{q}}$ simplifies to -1 for even p and the difference of two odds is even so $(-1)^{\frac{p-q}{q}}$ simplifies to 1 for odd p we can write,

$$\begin{aligned} \text{For even } p, \frac{d}{dx} (-x)^{\frac{p}{q}} &= \frac{p}{q} (-x)^{\frac{p-q}{q}} (-1) = \\ &= \frac{p}{q} (-1)^{\frac{p-q}{q}} x^{\frac{p-q}{q}} = \frac{p}{q} x^{\frac{p-q}{q}} \end{aligned}$$

$$\begin{aligned} \text{For odd } p, \frac{d}{dx} -(-x)^{\frac{p}{q}} &= -\frac{p}{q} (-x)^{\frac{p-q}{q}} (-1) = \\ &= \frac{p}{q} (-1)^{\frac{p-q}{q}} x^{\frac{p-q}{q}} = \frac{p}{q} x^{\frac{p-q}{q}} \end{aligned}$$

Rewriting in term of r we have for both even and odd p if $y = x^r$ then $\frac{dy}{dx} = r x^{r-1}$ ■

Derivatives of Logarithmic and Exponential Functions

$\frac{d}{dx}(e^x) = \lim_{h \rightarrow 0} \frac{e^{x+h} - e^x}{h}$ $= \lim_{h \rightarrow 0} \frac{e^x(e^h - 1)}{h}$ $= e^x \lim_{h \rightarrow 0} \frac{(e^h - 1)}{h}$ <p>Euler's number e can be defined by the limit: $e \triangleq \lim_{n \rightarrow \infty} (1 + 1/n)^n$.</p> <p>By substituting $h = 1/n$ this can be rewritten as $e = \lim_{h \rightarrow 0} (1 + h)^{1/h}$.</p> <p>Substituting this for e above in the right side gives,</p> $\frac{d}{dx}(e^x) = e^x \lim_{h \rightarrow 0} \frac{\left(\left(\lim_{h \rightarrow 0} (1 + h)^{1/h} \right)^h - 1 \right)}{h}$ $= e^x \lim_{h \rightarrow 0} \frac{\left((1 + h)^{h/h} - 1 \right)}{h}$ $= e^x \lim_{h \rightarrow 0} \frac{(1 + h - 1)}{h}$ $= e^x \lim_{h \rightarrow 0} \frac{h}{h}$ $= e^x \lim_{h \rightarrow 0} 1$ $= e^x$ <p>□</p>	<p>Use implicit differentiation:</p> <p>For positive x:</p> $y = \ln x$ $e^y = x$ $e^y \frac{dy}{dx} = 1$ $\frac{dy}{dx} = \frac{1}{e^y}$ $\frac{d}{dx}(\ln x) = \frac{1}{x}$ <p>□</p> <p>For negative x let:</p> $y = \ln(-x)$ $e^y = -x$ $e^y \frac{dy}{dx} = -1$ $\frac{dy}{dx} = \frac{-1}{e^y}$ $\frac{d}{dx}(\ln(-x)) = \frac{-1}{-x}$ $= \frac{1}{x}$ <p>Combined with the earlier result, $\frac{d}{dx}(\ln x) = \frac{1}{x}$. □</p>
$\frac{d}{dx}(b^x) = \frac{d}{dx}(e^{\ln b^x})$ $= \frac{d}{dx}(e^{x \ln b})$ $= e^{x \ln b} \ln b$ $= e^{\ln b^x} \ln b$ $= b^x \ln b$ <p>□</p>	$\frac{d}{dx}(\log_b x) = \frac{d}{dx} \left(\frac{\ln x}{\ln b} \right)$ $= \frac{1}{\ln b} \frac{d}{dx}(\ln x)$ $= \frac{1}{x \ln b}$ <p>□</p>

Derivatives of Trigonometric Functions

$\begin{aligned}\frac{d}{dx}(\sin x) &= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin x \cos h - \sin x}{h} + \lim_{h \rightarrow 0} \frac{\cos x \sin h}{h} \\ &= \sin x \cdot \lim_{h \rightarrow 0} \frac{(\cos h - 1)}{h} + \cos x \lim_{h \rightarrow 0} \frac{\sin h}{h} \\ &= \sin x \cdot 0 + (\cos x)1 \\ &= \cos x \\ &\square\end{aligned}$	$\begin{aligned}\frac{d}{dx}(\cos x) &= \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cos x \cos h - \cos x}{h} - \lim_{h \rightarrow 0} \frac{\sin x \sin h}{h} \\ &= \cos x \cdot \lim_{h \rightarrow 0} \frac{(\cos h - 1)}{h} - \sin x \lim_{h \rightarrow 0} \frac{\sin h}{h} \\ &= \cos x \cdot 0 - (\sin x)1 \\ &= -\sin x \\ &\square\end{aligned}$
$\begin{aligned}\frac{d}{dx}(\tan x) &= \frac{d}{dx} \left(\frac{\sin x}{\cos x} \right) \\ &= \frac{\cos x \cdot \frac{d}{dx} \sin x - \sin x \frac{d}{dx} \cos x}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} \\ &= \sec^2 x \\ &\square\end{aligned}$	$\begin{aligned}\frac{d}{dx}(\cot x) &= \frac{d}{dx} \left(\frac{\cos x}{\sin x} \right) \\ &= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} \\ &= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} \\ &= -\frac{1}{\sin^2 x} \\ &= -\csc^2 x \\ &\square\end{aligned}$
$\begin{aligned}\frac{d}{dx}(\sec x) &= \frac{d}{dx} \left(\frac{1}{\cos x} \right) \\ &= \frac{0 - (-\sin x)}{\cos^2 x} \\ &= \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} \\ &= \sec x \tan x \\ &\square\end{aligned}$	$\begin{aligned}\frac{d}{dx}(\csc x) &= \frac{d}{dx} \left(\frac{1}{\sin x} \right) \\ &= \frac{0 - \cos x}{\sin^2 x} \\ &= -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} \\ &= -\csc x \cot x \\ &\square\end{aligned}$

Derivatives of Inverse Trigonometric Functions

$y = \sin^{-1} x \rightarrow \cos^2 y = 1 - x^2$ $\sin y = x$ $\cos y \frac{dy}{dx} = 1$ $\frac{dy}{dx} = \frac{1}{\cos y}$ <p>cos y is always positive on the range of $\sin^{-1} x$, $y \in [-\pi/2, \pi/2]$. This allows us to place the denominator in absolute value signs.</p> $\frac{dy}{dx} = \frac{1}{ \cos y }$ $= \frac{1}{\sqrt{1-x^2}}$ <p>□</p>	$y = \cos^{-1} x \rightarrow \sin^2 y = 1 - x^2$ $\cos y = x$ $-\sin y \frac{dy}{dx} = 1$ $\frac{dy}{dx} = \frac{-1}{\sin y}$ <p>sin y is always positive on the range of $\cos^{-1} x$, $y \in [0, \pi]$. This allows us to place the denominator in absolute value signs.</p> $\frac{dy}{dx} = \frac{-1}{ \sin y }$ $= \frac{-1}{\sqrt{1-x^2}}$ <p>□</p>	$y = \tan^{-1} x \rightarrow \sec^2 y = x^2 + 1$ $\tan y = x$ $\sec^2 y \frac{dy}{dx} = 1$ $\frac{dy}{dx} = \frac{1}{\sec^2 y}$ $= \frac{1}{1+x^2}$ <p>□</p>
$y = \csc^{-1} x \rightarrow \cot^2 y = x^2 - 1$ $\csc y = x$ $-\csc y \cot y \frac{dy}{dx} = 1$ $\frac{dy}{dx} = \frac{-1}{\csc y \cot y}$ <p>The domain of $x = \csc y$ is $y \in [-\pi/2, 0) \cup (0, \pi/2]$. On this domain $\csc y$ and $\cot y$ are always the same sign so their product is always positive. This allows us to place the denominator in absolute value signs.</p> $\frac{dy}{dx} = \frac{-1}{ \csc y \cot y }$ $= \frac{-1}{ x \sqrt{x^2-1}}$ <p>□</p>	$y = \sec^{-1} x \rightarrow \tan^2 y = x^2 - 1$ $\sec y = x$ $\sec y \tan y \frac{dy}{dx} = 1$ $\frac{dy}{dx} = \frac{1}{\sec y \tan y}$ <p>The domain of $x = \sec y$ is $y \in [0, \pi/2) \cup (\pi/2, \pi]$. On this domain $\sec y$ and $\tan y$ are always the same sign so their product is always positive. This allows us to place the denominator in absolute value signs.</p> $\frac{dy}{dx} = \frac{1}{ \sec y \tan y }$ $= \frac{1}{ x \sqrt{x^2-1}}$ <p>□</p>	$y = \cot^{-1} x \rightarrow \csc^2 y = 1 + x^2$ $\cot y = x$ $-\csc^2 y \frac{dy}{dx} = 1$ $\frac{dy}{dx} = \frac{-1}{\csc^2 y}$ $= \frac{-1}{1+x^2}$ <p>□</p>

Proof of Integrals

Integral of Trigonometric Functions

$\int \tan x dx = \int \frac{\sin x}{\cos x} dx$ $= \int \frac{\sin x}{u} \frac{du}{-\sin x}; u = \cos x \rightarrow du = -\sin x dx$ $= -\int \frac{1}{u} du$ $= -\ln u $ $= -\ln \cos x + C$ $= \ln \sec x + C$ <p style="text-align: center;">□</p>	$\int \cot x dx = \int \frac{\cos x}{\sin x} dx$ $= \int \frac{\cos x}{u} \frac{du}{\cos x}; u = \sin x \rightarrow du = \cos x dx$ $= \int \frac{1}{u} du$ $= \ln u $ $= \ln \sin x + C$ <p style="text-align: center;">□</p>
$\int \csc x dx = \int \frac{1}{\sin x} dx$ $= \int \frac{\sin x}{\sin^2 x} dx$ $= \int \frac{\sin x}{1 - \cos^2 x} dx$ $= \int \frac{\sin x}{1 - u^2} \frac{-du}{\sin x}; u = \cos x \rightarrow du = -\sin x dx$ $= \int \frac{1}{u^2 - 1} du$ $= \int \left(\frac{-1/2}{u+1} - \frac{-1/2}{u-1} \right) du; \text{ using partial fraction expansion}$ $= -\frac{1}{2} \left(\int \frac{du}{u+1} - \int \frac{du}{u-1} \right)$ $= -\frac{1}{2} (\ln u+1 - \ln u-1)$ $= \frac{1}{2} \ln \left \frac{u-1}{u+1} \right $ $= \frac{1}{2} \ln \left \frac{u-1}{u+1} \frac{u-1}{u-1} \right $ $= \frac{1}{2} \ln \left \frac{(u-1)^2}{u^2-1} \right $ $= \frac{1}{2} \ln \left \frac{(\cos x - 1)^2}{\cos^2 x - 1} \right $ $= \frac{1}{2} \ln \left \frac{(\cos x - 1)^2}{-\sin^2 x} \right $ $= \ln \left(\frac{(\cos x - 1)^2}{\sin^2 x} \right)^{1/2}; \text{ using } -\sin^2 x = \sin^2 x$ $= \ln \left \frac{\cos x - 1}{\sin x} \right ; \text{ use } \text{ to maintain positive root}$ $= \ln \cot x - \csc x + C = \ln \csc x - \cot x + C$ <p style="text-align: center;">□</p>	$\int \sec x dx = \int \frac{1}{\cos x} dx$ $= \int \frac{\cos x}{\cos^2 x} dx$ $= \int \frac{\cos x}{1 - \sin^2 x} dx$ $= \int \frac{\cos x}{1 - u^2} \frac{du}{\cos x}; u = \sin x \rightarrow du = \cos x dx$ $= \int \frac{1}{1 - u^2} du$ $= \int \left(\frac{1/2}{1+u} + \frac{1/2}{1-u} \right) du; \text{ using partial fraction expansion}$ $= \frac{1}{2} \left(\int \frac{du}{1+u} + \int \frac{du}{1-u} \right)$ $= \frac{1}{2} (\ln 1+u - \ln 1-u)$ $= \frac{1}{2} \ln \left \frac{1+u}{1-u} \right $ $= \frac{1}{2} \ln \left \frac{1+u}{1-u} \frac{1+u}{1+u} \right $ $= \frac{1}{2} \ln \left \frac{(1+u)^2}{1-u^2} \right $ $= \frac{1}{2} \ln \left \frac{(1+\sin x)^2}{1-\sin^2 x} \right $ $= \frac{1}{2} \ln \left \frac{(1+\sin x)^2}{\cos^2 x} \right $ $= \ln \left(\frac{(1+\sin x)^2}{\cos^2 x} \right)^{1/2}$ $= \ln \left \frac{1+\sin x}{\cos x} \right ; \text{ use } \text{ to maintain positive root}$ $= \ln \sec x + \tan x + C$ <p style="text-align: center;">□</p>

Integral of Inverse Trigonometric Functions

$\int \sin^{-1} x dx = \int u dv = uv - \int v du$ <p>where $u = \sin^{-1} x$ and $dv = dx$</p> <p>so that $du = \frac{1}{\sqrt{1-x^2}} dx$ and $v = x$</p> $\int \sin^{-1} x dx = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx$ <p>let $w = 1-x^2 \rightarrow dw = -2x dx$</p> $\int \sin^{-1} x dx = x \sin^{-1} x - \int \frac{x}{-2x\sqrt{w}} dw;$ $= x \sin^{-1} x + \frac{1}{2} \int \frac{1}{\sqrt{w}} dw$ $= x \sin^{-1} x + \frac{1}{2} 2\sqrt{w}$ $= x \sin^{-1} x + \sqrt{1-x^2} + C$ <p>□</p>	$\int \csc^{-1} x dx = \int u dv = uv - \int v du; \quad u = \csc^{-1} x, \quad dv = dx \rightarrow du = \frac{-1}{x\sqrt{x^2-1}} dx, \quad v = x$ $= x \csc^{-1} x + \int \frac{x}{x\sqrt{x^2-1}} dx$ $= x \csc^{-1} x + \int \frac{1}{\sqrt{x^2-1}} dx$ <p>Let $x = \csc u$ so $dx = -\csc u \cot u du$ and $\cot u = \sqrt{x^2-1}$</p> $= x \csc^{-1} x + \int \frac{-\csc u \cot u}{\sqrt{\csc^2 u - 1}} du$ $= x \csc^{-1} x - \int \frac{\csc u \cot u}{\cot u} du$ $= x \csc^{-1} x - \int \csc u du$ $= x \csc^{-1} x + \ln \cot u + \csc u $ $= x \csc^{-1} x + \ln \left x + \sqrt{x^2-1} \right $ <p>□</p>
$\int \cos^{-1} x dx = \int u dv = uv - \int v du$ <p>where $u = \cos^{-1} x$ and $dv = dx$</p> <p>so that $du = \frac{-1}{\sqrt{1-x^2}} dx$ and $v = x$</p> $\int \cos^{-1} x dx = x \cos^{-1} x + \int \frac{x}{\sqrt{1-x^2}} dx$ <p>see $\int \sin^{-1} x dx$ for solution to $\int \frac{x}{\sqrt{1-x^2}} dx$</p> $\int \cos^{-1} x dx = x \cos^{-1} x - \sqrt{1-x^2} + C$ <p>□</p>	$\int \sec^{-1} x dx = \int u dv = uv - \int v du; \quad u = \sec^{-1} x, \quad dv = dx \rightarrow du = \frac{1}{x\sqrt{x^2-1}} dx, \quad v = x$ $= x \sec^{-1} x - \int \frac{x}{x\sqrt{x^2-1}} dx$ $= x \sec^{-1} x - \int \frac{1}{\sqrt{x^2-1}} dx$ $= x \sec^{-1} x - \int \frac{\sec u \tan u}{\sqrt{\sec^2 u - 1}} du; \quad x = \sec u \rightarrow dx = \sec u \tan u du$ $= x \sec^{-1} x - \int \frac{\sec u \tan u}{\tan u} du$ $= x \sec^{-1} x - \int \sec u du$ $= x \sec^{-1} x - \ln \sec u + \tan u ; \quad x = \sec u \rightarrow \tan u = \sqrt{x^2-1}$ $= x \sec^{-1} x - \ln \left x + \sqrt{x^2-1} \right $ <p>□</p>
$\int \tan^{-1} x dx = \int u dv = uv - \int v du;$ <p>where $u = \tan^{-1} x$ and $dv = dx$</p> <p>so that $du = \frac{1}{1+x^2} dx$ and $v = x$</p> $\int \tan^{-1} x dx = x \tan^{-1} x - \int \frac{x}{1+x^2} dx$ <p>Let $w = 1+x^2 \rightarrow dw = 2x dx$</p> $\int \tan^{-1} x dx = x \tan^{-1} x - \int \frac{x}{2xw} dw$ $= x \tan^{-1} x - \frac{1}{2} \int \frac{1}{w} dw$ $= x \tan^{-1} x - \frac{1}{2} \ln(1+x^2)$ $= x \tan^{-1} x - \ln \sqrt{1+x^2} + C$ <p>□</p>	$\int \cot^{-1} x dx = \int u dv = uv - \int v du; \quad u = \cot^{-1} x, \quad dv = dx \rightarrow du = \frac{-1}{1+x^2} dx, \quad v = x$ $= x \cot^{-1} x + \int \frac{x}{1+x^2} dx, \text{ see } \int \tan^{-1} x dx \text{ for details on solving this integral}$ $= x \cot^{-1} x + \ln \sqrt{1+x^2} + C$ <p>□</p>

$$\int \frac{1}{\sqrt{x^2 \pm a^2}} dx \quad \text{and} \quad \int \frac{1}{\sqrt{a^2 - x^2}} dx$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx$$

Let $x = a \csc u \rightarrow dx = -a \csc u \cot u du$
and $\cot u = \sqrt{x^2 - a^2}/a$, $u \in [-\pi/2, 0) \cup (0, \pi/2]$

$$\begin{aligned} \int \frac{1}{\sqrt{x^2 - a^2}} dx &= \int \frac{1}{a\sqrt{\csc^2 u - 1}} - a \csc u \cot u du \\ &= -\int \frac{1}{\cot u} \csc u \cot u du \\ &= -\int \csc u du \\ &= -\ln|\csc u - \cot u| + C_0 \\ &= -\ln \left| \frac{x}{a} - \frac{\sqrt{x^2 - a^2}}{a} \right| + C_0 \\ &= -\ln \left| \frac{x - \sqrt{x^2 - a^2}}{a} \cdot \frac{x + \sqrt{x^2 - a^2}}{x + \sqrt{x^2 - a^2}} \right| + C_0 \\ &= -\ln \left| \frac{a}{x + \sqrt{x^2 - a^2}} \right| + C_0 \\ &= \ln|x + \sqrt{x^2 - a^2}| - \ln a + C_0 \\ &= \ln|x + \sqrt{x^2 - a^2}| + C \end{aligned}$$

□

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx$$

Let $x = a \tan u \rightarrow dx = a \sec^2 u du$ and $\sec u = \sqrt{x^2 + a^2}/a$,
 $u \in [0, \pi/2) \cup (\pi/2, \pi]$

$$\begin{aligned} \int \frac{1}{\sqrt{x^2 + a^2}} dx &= \int \frac{1}{a\sqrt{\tan^2 u + 1}} a \sec^2 u du \\ &= \int \frac{1}{\sec u} \sec^2 u du \\ &= \int \sec u du \\ &= \ln|\sec u + \tan u| + C_0 \\ &= \ln \left| \frac{\sqrt{x^2 + a^2} + x}{a} \right| + C_0 \\ &= \ln|\sqrt{x^2 + a^2} + x| - \ln a + C_0 \\ &= \ln|\sqrt{x^2 + a^2} + x| + C \\ &= \ln(\sqrt{x^2 + a^2} + x) + C \end{aligned}$$

□

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx$$

Let $u = x/a \rightarrow x = au$ and $dx = a du$

$$\begin{aligned} \int \frac{1}{\sqrt{a^2 - x^2}} dx &= \int \frac{a}{\sqrt{a^2 - a^2 u^2}} du \\ &= \int \frac{1}{\sqrt{1 - u^2}} du \\ &= \sin^{-1} u + C \\ &= \sin^{-1} x/a + C \end{aligned}$$

Since $\sin^{-1} \theta = \pi/2 - \cos^{-1} \theta$,

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} x/a + C = -\cos^{-1} x/a + C$$

□

$$\int \frac{1}{x^2 \pm a^2} dx, \int \sqrt{x^2 - a^2} dx \quad \text{and} \quad \int \sqrt{a^2 - x^2} dx$$

$\int \frac{1}{x^2 - a^2} dx$ <p>Use the partial fraction expansion $\frac{1}{x^2 - a^2} = \frac{1/(2a)}{x - a} + \frac{-1/(2a)}{x + a}$.</p> $\int \frac{1}{x^2 - 1} dx = \frac{1}{2a} \left(\int \frac{1}{x - a} dx - \int \frac{1}{x + a} dx \right)$ $= \frac{1}{2a} (\ln x - a - \ln x + a) + C$ $= \frac{1}{2a} \ln \left \frac{x - a}{x + a} \right + C$ <p>□</p>	$\int \frac{1}{x^2 + a^2} dx$ <p>Let $u = x/a \rightarrow x = au$ and $dx = a du$</p> $\int \frac{1}{x^2 + a^2} dx = \int \frac{a}{a^2 u^2 + a^2} du$ $= \frac{1}{a} \int \frac{1}{u^2 + 1} du$ $= \frac{1}{a} \tan^{-1} u + C$ $= \frac{1}{a} \tan^{-1} x/a + C$ <p>Since $\tan^{-1} \theta = \pi/2 - \cot^{-1} \theta$,</p> $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \frac{1}{a} \tan^{-1} x/a + C = -\frac{1}{a} \cot^{-1} x/a + C$ <p>□</p>
$\int \sqrt{x^2 - a^2} dx$ <p>Let $x = a \sec u \rightarrow dx = a \sec u \tan u du$ and $\tan u = \frac{\sqrt{x^2 - a^2}}{a}$.</p> $\int \sqrt{x^2 - a^2} dx = \int \sqrt{a^2 \sec^2 u - a^2} (a \sec u \tan u) du$ $= \int (a \tan u) a \sec u \tan u du$ $= a^2 \int \sec u \tan^2 u du$ <p>Rewrite the integrand using the Pythagorean identity,</p> $\sec u \tan^2 u = \frac{\sin^2 u}{\cos^3 u} = \frac{1 - \cos^2 u}{\cos^3 u} = \sec^3 u - \sec u$ $\int \sqrt{x^2 - a^2} dx = a^2 \left(\int \sec^3 u du - \int \sec u du \right)$ $= a^2 \left(\frac{1}{2} \sec u \tan u + \frac{1}{2} \ln \sec u + \tan u - \ln \sec u + \tan u \right) + C_0$ $= \frac{a^2}{2} \sec u \tan u - \frac{1}{2} a^2 \ln \sec u + \tan u + C_0$ $= \frac{a^2}{2} \frac{x}{a} \frac{\sqrt{x^2 - a^2}}{a} - \frac{1}{2} a^2 \ln \left \frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a} \right + C_0$ $= \frac{x}{2} \sqrt{x^2 - a^2} - \frac{1}{2} a^2 \ln x + \sqrt{x^2 - a^2} + \frac{3}{2} a^2 \ln a + C_0$ $= \frac{x}{2} \sqrt{x^2 - a^2} - \frac{1}{2} a^2 \ln x + \sqrt{x^2 - a^2} + C$ <p>□</p>	$\int \sqrt{a^2 - x^2} dx$ <p>Let $x = a \sin u \rightarrow dx = a \cos u du$. This means</p> $u = \sin^{-1} \frac{x}{a},$ $\sin u = \frac{x}{a} \quad \text{and} \quad \cos u = \frac{\sqrt{a^2 - x^2}}{a}$ $\int \sqrt{a^2 - x^2} dx = \int \sqrt{a^2 - a^2 \sin^2 u} (a \cos u) du$ $= \int \sqrt{a^2 \cos^2 u} (a \cos u) du$ $= a^2 \int \cos^2 u du$ <p>Use identity $\cos^2 u = \frac{1 + \cos 2u}{2}$,</p> $= \frac{a^2}{2} \int du + \frac{a^2}{2} \int \cos 2u du$ $= \frac{a^2}{2} u + \frac{a^2}{4} \sin 2u$ <p>Use identity $\sin 2u = 2 \sin u \cos u$,</p> $= \frac{a^2}{2} u + \frac{a^2}{4} 2 \sin u \cos u$ $= \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{a^2}{2} \left(\frac{x}{a} \right) \left(\frac{\sqrt{a^2 - x^2}}{a} \right)$ $= \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{x}{2} \sqrt{a^2 - x^2} + C$ <p>□</p>

Integral Reduction of Powers of Trigonometric Functions

$\int \sin^n x dx = \int \sin^{n-1} x \sin x dx$ <p>Let $u = \sin^{n-1} x$ and $dv = \sin x dx \rightarrow du = (n-1)\sin^{n-2} x \cos x dx$ and $v = -\cos x$</p> $\int \sin^n x dx = \int u dv$ $= uv - \int v du$ $= -\cos x \sin^{n-1} x + \int \cos x (n-1)\sin^{n-2} x \cos x dx$ $= -\cos x \sin^{n-1} x + \int \cos^2 x (n-1)\sin^{n-2} x dx$ $= -\cos x \sin^{n-1} x + \int (1 - \sin^2 x)(n-1)\sin^{n-2} x dx$ $= -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x dx - (n-1) \int \sin^n x dx$ $n \int \sin^n x dx = -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x dx$ $\int \sin^n x dx = \frac{-1}{n} \cos x \sin^{n-1} x + \frac{(n-1)}{n} \int \sin^{n-2} x dx \quad \square$	$\int \cos^n x dx = \int \cos^{n-1} x \cos x dx$ <p>Let $u = \cos^{n-1} x$ and $dv = \cos x dx \rightarrow du = -(n-1)\cos^{n-2} x \sin x dx$ and $v = \sin x$</p> $\int \cos^n x dx = \int u dv$ $= uv - \int v du$ $= \cos^{n-1} x \sin x + (n-1) \int \sin x \cos^{n-2} x \sin x dx$ $= \cos^{n-1} x \sin x + (n-1) \int \sin^2 x \cos^{n-2} x dx$ $= \cos^{n-1} x \sin x + (n-1) \int (1 - \cos^2 x) \cos^{n-2} x dx$ $= \cos^{n-1} x \sin x + (n-1) \cos^{n-2} x dx - (n-1) \int \cos^n x dx$ $n \int \cos^n x dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x dx$ $\int \cos^n x dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x dx \quad \square$
$\int \tan^n x dx = \int \tan^{n-2} x \tan^2 x dx$ $= \int \tan^{n-2} x (\sec^2 x - 1) dx$ $= \int \tan^{n-2} x \sec^2 x dx - \int \tan^{n-2} x dx$ <p>Let $u = \tan^{n-2} x$ and $dv = \sec^2 x dx \rightarrow du = (n-2)\tan^{n-3} x \sec^2 x dx$ and $v = \tan x$</p> $\int \tan^{n-2} x \sec^2 x dx = \int u dv$ $= uv - \int v du$ $= \tan^{n-1} x - \int \tan x (n-2)\tan^{n-3} x \sec^2 x dx$ $= \tan^{n-1} x - (n-2) \int \tan^{n-2} x \sec^2 x dx$ $(n-1) \int \tan^{n-2} x \sec^2 x dx = \tan^{n-1} x$ $\int \tan^{n-2} x \sec^2 x dx = \frac{1}{n-1} \tan^{n-1} x$ $\int \tan^n x dx = \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x dx \quad \square$	$\int \cot^n x dx = \int \cot^{n-2} x \cot^2 x dx$ $= \int \cot^{n-2} x (\csc^2 x - 1) dx$ $= \int \cot^{n-2} x \csc^2 x dx - \int \cot^{n-2} x dx$ <p>Let $u = \cot^{n-2} x$ and $dv = \csc^2 x dx \rightarrow du = -(n-2)\cot^{n-3} x \csc^2 x dx$ and $v = -\cot x$</p> $\int \cot^{n-2} x \csc^2 x dx = \int u dv$ $= uv - \int v du$ $= -\cot^{n-1} x - \int \cot x (n-2)\cot^{n-3} x \csc^2 x dx$ $= -\cot^{n-1} x - (n-2) \int \cot^{n-2} x \csc^2 x dx$ $(n-1) \int \cot^{n-2} x \csc^2 x dx = -\cot^{n-1} x$ $\int \cot^{n-2} x \csc^2 x dx = \frac{-1}{n-1} \cot^{n-1} x$ $\int \cot^n x dx = \frac{-1}{n-1} \cot^{n-1} x - \int \cot^{n-2} x dx \quad \square$
$\int \csc^n x dx = \int \csc^{n-2} x \csc^2 x dx$ <p>Let $u = \csc^{n-2} x$ and $dv = \csc^2 x dx \rightarrow du = -(n-2)\csc^{n-2} x \cot x dx$ and $v = -\cot x$</p> $\int \csc^n x dx = \int u dv$ $= uv - \int v du$ $= -\csc^{n-2} x \cot x - (n-2) \int \cot^2 x \csc^{n-2} x dx$ $= -\csc^{n-2} x \cot x - (n-2) \int (\csc^2 - 1) \csc^{n-2} x dx$ $= -\csc^{n-2} x \cot x - (n-2) \left(\int \csc^n x dx - \int \csc^{n-2} x dx \right)$ $= -\csc^{n-2} x \cot x - (n-2) \int \csc^n x dx + (n-2) \int \csc^{n-2} x dx$ $(n-1) \int \csc^n x dx = -\csc^{n-2} x \cot x + (n-2) \int \csc^{n-2} x dx$ $\int \csc^n x dx = \frac{-1}{n-1} \csc^{n-2} x \cot x + \frac{n-2}{n-1} \int \csc^{n-2} x dx \quad \square$	$\int \sec^n x dx = \int \sec^{n-2} x \sec^2 x dx$ <p>Let $u = \sec^{n-2} x$ and $dv = \sec^2 x dx \rightarrow du = (n-2)\sec^{n-2} x \tan x dx$ and $v = \tan x$</p> $\int \sec^n x dx = \int u dv$ $= uv - \int v du$ $= \sec^{n-2} x \tan x - (n-2) \int \tan^2 x \sec^{n-2} x dx$ $= \sec^{n-2} x \tan x - (n-2) \int (\sec^2 - 1) \sec^{n-2} x dx$ $= \sec^{n-2} x \tan x - (n-2) \left(\int \sec^n x dx - \int \sec^{n-2} x dx \right)$ $= \sec^{n-2} x \tan x - (n-2) \int \sec^n x dx + (n-2) \int \sec^{n-2} x dx$ $(n-1) \int \sec^n x dx = \sec^{n-2} x \tan x + (n-2) \int \sec^{n-2} x dx$ $\int \sec^n x dx = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x dx \quad \square$

Derivative, Area, and Integral Relationship

An intuitive explanation of the relationship between area, the derivative, and anti-derivative or integral is given below.

We are given a function f that is continuous on some interval. Let F describe the area of the region bounded between f and the x -axis over the interval.

Choose a point x and a nearby point $x + h$ in the interval. The area of this region is then $F(x + h) - F(x)$.

This area can also be approximated as $f(x + h) \cdot h$ for small h so,

$$F(x + h) - F(x) \approx f(x + h) \cdot h$$
$$\frac{F(x + h) - F(x)}{h} \approx f(x + h).$$

If we take $\lim_{h \rightarrow 0}$ the approximation becomes exact, the left side becomes the $F'(x)$ and the right side $f(x)$: $F'(x) = f(x)$.

This shows that the derivative of the function that describes the area under a curve is the function that describes that curve.