

Important Definitions and Theorems

Definitions

Limit

Given a function f defined in an open interval containing the number a , except that f need not be defined at a , if for any number $\varepsilon > 0$ there exists a number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \text{ when } 0 < |x - a| < \delta,$$

then the limit of f as x approaches a equals L , written $\lim_{x \rightarrow a} f(x) = L$.

Continuity

A function f is continuous at c if all the following are true:

- 1) $f(c)$ is defined
- 2) $\lim_{x \rightarrow c} f(x)$ exists
- 3) $\lim_{x \rightarrow c} f(x) = f(c)$.

Increasing, Decreasing and Constant Functions

- 1) f is **increasing*** on an interval I if $f(b) > f(a)$ for all a and b in I where $b > a$.
- 2) f is **decreasing*** on an interval I if $f(b) < f(a)$ for all a and b in I where $b > a$.
- 3) f is **constant** on an interval I if $f(a) = f(b)$ for all a and b in I .

*Some texts call 1 and 2 above strictly increasing and strictly decreasing and use $f(b) \geq f(a)$ for increasing and $f(b) \leq f(a)$ for decreasing. With this definition a constant function is both increasing and decreasing.

Critical & Stationary Points

If f is a function defined on an open interval containing c and $f'(c)$ is 1) zero or 2) undefined, then c is called a **critical point** of f . In case 1) we also call c a **stationary point**. These values of c are candidates for relative extremum.

Local (Relative) Extrema

- 1) f has a **local maximum*** at c if there exists an open interval I containing c where $f(c) \geq f(x)$ for all x in I .
- 2) f has a **local minimum*** at c if there exists an open interval I containing c where $f(c) \leq f(x)$ for all x in I .

*Some authors allow c to be the endpoint of a closed domain. In this case I is a half-open interval. This allows endpoints to be candidates for local extrema which are excluded in the definition above by the requirement that I to be an open interval.

Absolute (Global) Extrema

- 1) f has an **absolute maximum** at c in interval I if $f(c) \geq f(x)$ for all x in I .
- 2) f has an **absolute minimum** at c in interval I if $f(c) \leq f(x)$ for all x in I .

Concavity

If f is differentiable on an open interval I , then

- 1) f is **concave up** on I if f' is increasing on I .
- 2) f is **concave down** on I if f' is decreasing on I .

Point of Inflection

If f is continuous on an open interval containing c and if f changes the direction of its concavity at point $(c, f(c))$ then f is said to have an **inflection point at c** and the point $(c, f(c))$ on the graph of f is an **inflection point** of f .

- If the point occurs where the derivative is infinite it is called a **vertical inflection point**.
- If the point occurs where the derivative is zero it is called a **stationary inflection point** or **horizontal inflection point**.

Cusps and Corners

Cusps and **corners** occur on a *continuous* interval at a point where the first derivative does not exist *and* the left and right sided derivatives are distinct.

Theorems

Intermediate Value Theorem

If function f is *continuous* on an interval $[a, b]$ and $f(a) \leq L \leq f(b)$ then there exists a value c in $[a, b]$ such that $f(c) = L$.

In other words, f will take on every value between $f(a)$ and $f(b)$ over the interval $[a, b]$.

Extreme Value Theorem

If function f is continuous on an interval $[a, b]$ then there exists both an absolute maximum and absolute minimum of f on $[a, b]$.

Rolle's Theorem

If function f is continuous on an interval $[a, b]$ and differentiable on the interval (a, b) and $f(a) = f(b) = 0$ then there exists a value c in (a, b) such that $f'(c) = 0$.

Mean-Value Theorem

If function f is continuous on an interval $[a, b]$ and differentiable on the interval (a, b) then there exists a value c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Mean-Value Theorem for Integrals

If function f is continuous on an interval $[a, b]$ then there exists a value c in (a, b) such

$$f(c) = \frac{1}{b - a} \int_a^b f(x) dx \triangleq f_{av}.$$

The value f_{av} is defined as the average value of the function f on the interval $[a, b]$.

Fundamental Theorem of Calculus

- i. If function f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then $\int_a^b f(x) dx = F(b) - F(a)$.
- ii. If function f is continuous on $[a, b]$ then for any constant c in $[a, b]$ the function $F(x) \triangleq \int_c^x f(t) dt$ $x \in [a, b]$ is an antiderivative of f and F is continuous on $[a, b]$, differentiable on (a, b) and $F'(x) = f(x)$.

A geometric explanation of the relationship between area, the derivative, and anti-derivative (integral): Given a function f that is continuous on some interval, let F describe the area of the region bounded between f and the x -axis over the interval. Chose a point x and a nearby point $x + h$ in the interval. The area of this region is then $F(x + h) - F(x)$. This area can also be approximated as $f(x + h) \cdot h$ for small h so,

$$\begin{aligned} F(x + h) - F(x) &\approx f(x + h) \cdot h \\ \frac{F(x + h) - F(x)}{h} &\approx f(x + h). \end{aligned}$$

If we take $\lim_{h \rightarrow 0}$ the approximation becomes exact, the left side becomes the $F'(x)$ and the right side $f(x)$, thus $F'(x) = f(x)$. This shows that the derivative of the function that describes the area under a curve is the function that describes that curve.

One-to-One Theorem

If the domain of function f is an open interval and its derivative is either always positive or always negative, then f is one-to-one and it follows that its inverse is differentiable for all values in the *range* of f (which is the domain of f^{-1}).

Derivative of Inverse Function Theorem

Given a function f that has an inverse $f^{-1} = g$, if f is differentiable at $g(x)$ and $f'(g(x)) \neq 0$ then $g'(x) = \frac{1}{f'(g(x))}$.

If we call $y = f^{-1}(x) \rightarrow x = f(y)$ this can be expressed as $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$.

In other words, the slopes of the graphs of f and f^{-1} are multiplicative inverses (where they are both defined). To find the slope of f^{-1} at a solve $f(b) = a$ for b and then the slope is $(f^{-1})'(a) = \frac{1}{f'(b)}$, where $a = f(b)$.